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p-compact groups as framed manifolds

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Abstract

We describe a natural way to associate to any p-compact group an element of the p-local stable stems, which, applied to the p-completion of a compact Lie group G, coincides with the element represented by the manifold G with its left-invariant framing. To this end, we construct a d-dimensional sphere S_G with a stable G-action for every d-dimensional p-compact group G, which generalizes the one-point compactification of the Lie algebra of a Lie group. The homotopy class represented by G is then constructed by means of a transfer map between the Thom spaces of spherical fibrations over BG associated with S_G.

Keywords: p-compact groups; Pontryagin–Thom construction; Adjoint representation; Transfer

1. Introduction

Let G be a compact connected Lie group with (real) Lie algebra $\mathfrak{g} = T_eG$. Left multiplication with an element $g \in G$ gives an isomorphism $\mathfrak{g} \cong T_gG$, and by choosing a basis for $\mathfrak{g}$, we thus obtain a framing L of the manifold G, called the left-invariant framing. By Stong [28, p. 23f], this is equivalent to a trivialization of the stable normal bundle, and the Pontryagin–Thom construction produces from this data an element in $\pi_d^s(\mathbf{S}^0)$, where $d = \dim G$. Computations of homotopy classes that arise in this way have been made by Smith [27], Wood [30], Knapp [17], Becker and Schultz [6], and others. The most extensive table of homotopy classes represented by Lie groups to date can be found in Ossa [23]. A general formula is not known, even for the case of any family of classical simple Lie groups.

The Pontryagin–Thom construction is intimately related to the transfer map for the universal bundle over the classifying space of the Lie group G. More generally, for every subgroup inclusion $H \overset{i}{\hookrightarrow} G$

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of Lie groups, there is a transfer map in the stable homotopy category $\Sigma^\infty BG^{\mathfrak{g}} \xrightarrow{i_!} \Sigma^\infty BH^{\mathfrak{h}}$. Here $BG^{\mathfrak{g}}$ stands for the Thom space of the bundle associated to the adjoint representation of G on $\mathfrak{g}$. This map is a twisted version of the well-known Umkehr map for the fibration $G/H \to BH \xrightarrow{Bi} BG$,

$$BG_+ \to BH^{\nu}, \tag{1.1}$$

where ν stands for the normal bundle along the fibers of p. Note that the tangent bundle along the fibers of p is $\mathfrak{g}/\mathfrak{h}$ and hence $\nu = \mathfrak{h} - \mathfrak{g}$ as virtual vector bundles. By taking Thom spaces with respect to the bundle $\mathfrak{g}$ resp. $p^*\mathfrak{g}$ on both sides in (1.1), we obtain the desired map.

Lemma 1. *The homotopy class represented by the d-dimensional compact Lie group G is given by the following composite of maps* (*up to a sign*):

$$\mathbf{S}^d \to \Sigma^\infty BG^{\mathfrak{g}} \xrightarrow{i_!} \Sigma^\infty EG_+ \simeq \mathbf{S}^0.$$

Here the left-hand map is the inclusion of the bottom cell into $BG^{\mathfrak{g}}$, and i is the inclusion of the trivial subgroup into G.

Note that we can factor this map through any $BH^{\mathfrak{h}}$, where $H < G$. For $H = T$ a maximal torus in a semisimple G, this leads to an explicit way of computing the corresponding element in π_d^s.

In this paper, we go one step further and show that the transfer functor $(-)_!$ can be extended to the class of all *p-compact groups*. A p-compact group [12] G is an $H_*(-;\mathbf{Z}/p)$-local space BG such that $G =_{\mathrm{def}} \Omega BG$ has totally finite mod-p homology. Prominent examples are given by $H\mathbf{Z}/p$-localizations of compact Lie groups whose group of components are finite p-groups. Dwyer and Wilkerson have worked out an extensive Lie theory of p-compact groups [12]. It turns out that the classification of p-compact groups, at least at odd primes, is closely related to the classical classification of complex reflection groups by Sheppard and Todd [26], refined by Clark and Ewing [8] to p-adic reflection groups. These groups occur as "Weyl groups" of p-compact groups, and the p-compact groups themselves have been constructed on a case-by-case basis; no general method to construct them from their Weyl groups is known so far.

The main results of this paper are

Theorem 2. (1) *For every connected p-compact group G of $\mathbf{F}_p$-homological dimension d, there is a $H\mathbf{Z}/p$-local d-dimensional sphere S_G with a stable G-action, which in the case of the localization of a compact Lie group is equivalent to the localization of the one-point compactification of the Lie algebra $\mathfrak{g}$ with the adjoint action.*

(2) *For every monomorphism $H < G$ of p-compact groups, there is a map $S_G \to G_+ \wedge_H S_H$ which is an isomorphism in $H_d(-;\mathbf{F}_p)$.*

Annoyingly, the morphism in (2) fails to be G-equivariant, but it does so in a well-behaved manner. In fact, there is an extension to $EG_+ \wedge S_G \to G_+ \wedge_H S_H$ that is G-equivariant.

Using this "stable adjoint representation" S_G, we derive Spanier–Whitehead self-duality for p-compact groups:

Theorem 3. *For any monomorphism $H \hookrightarrow G$ of connected p-compact groups, there is a homotopy equivalence*

$$G_+ \wedge_H S_H \simeq D(G/H_+) \wedge S_G.$$

In the case where H is a torus, this shows that $\Sigma^\infty(G/H)_+ \simeq \Sigma^{d-d'}D(G/H)_+$, where d is the dimension of G and d' is the dimension of H. In particular, G/H is stably reducible, which is used in [5] to show that every finite loop space is homotopy equivalent to a compact, smooth manifold.

Theorem 4. *There is a contravariant functor t from the category of connected p-compact groups and monomorphisms to the stable homotopy category with the following properties*:

(1) *The spectrum*

$$BG^{\mathfrak{g}} := t(G) = EG_+ \wedge_G S_G$$

is $H\mathbf{Z}/p$-local and connective, and $H^*(BG^{\mathfrak{g}}; \mathbf{Z}_p)$ is a free module over $H^*(BG; \mathbf{Z}_p)$ on a Thom class in dimension d, the dimension of G.

(2) *The functor t makes the following diagram commute*:

$$\begin{tikzcd} S_G \arrow[r] \arrow[d] & G_+ \wedge_H S_G \arrow[d] \\ BG^{\mathfrak{g}} \arrow[r] & BH^{\mathfrak{h}} \end{tikzcd}$$

(3) *The composition $t \circ L_p$, defined on the category of compact Lie groups and monomorphisms (where L_p is $H\mathbf{Z}/p$ localization), is equivalent to the functor $L_p \circ (-)_!$.*

Theorem 4 enables us, by means of Lemma 1, to associate to any p-compact group an element in the stable stems, which one might provocatively call "the p-compact group in its invariant framing".

Table 1 shows a list of simple p-compact groups, ordered by their classification number in Sheppard and Todd's list, along with some known values of the homotopy classes they represent.

1.1. Outlook

The sphere S_G constructed in Theorem 2 has only a stable action by the p-compact group G. While not useful for the purposes of this paper, it would be interesting to see whether this action can be destabilized to yield an actual G-action on the space level.

A question that remains largely open is a philosophical one: what is it, from a homotopy point of view, that singles out the homotopy classes represented by Lie groups (or p-compact groups)? Hopkins and Mahowald conjectured that they are all detected in the Hopkins–Miller theory eo_2 of topological modular forms [13,14], i.e., if $[G] \neq 0 \in \pi_*^s$ then the Hurewicz image of $[G]$ in $(eo_2)_*$ is also nonzero. This is true for all simple Lie groups whose images in the stable stems are known, but it fails for example for the Lie group $\mathrm{Sp}(2) \times \mathrm{Sp}(2) \times \mathrm{SU}(2)$, which represents $\beta_1^2\alpha_1$ at prime 3, a class not detected by eo_2.[1] For p-compact groups, $p > 2$, the appropriate generalization of this conjecture would be that all simple p-compact groups are detected in the theory eo_{p-1}. This holds for the (few) examples computed in this paper.

Becker and Schultz conjecture in [6] that almost all compact Lie group represent trivial homotopy classes. We conjecture that this is also true for p-compact groups for every fixed p. Both conjectures

[1] However, Mahowald [19] conjectures they can all still be detected using a construction involving the Brown–Comenetz dual of eo_2.

Table 1
Simply connected, simple p-compact groups and the homotopy classes they represent

Name	Dimension	Rank	ST number	Prime	Homotopy class
$\mathbf{A_n}$	$n(n+2)$	n	1	Any	ν, ε, $\kappa\eta$, ?
$\mathbf{X}(m,q,n)$ [22]	$m(n^2-n+2(n/q))-n$	n	$2a$	1 (m)	? (some known)
I_{2m}	$2m+2$	2	$2b$	± 1 (m)	0
μ_m	$2m-1$	1	3	1 (m)	α_1 for $m=p-1$
	18	2	4	1 (3)	0
	34	2	5	1 (3)	0
	30	2	6	1 (12)	0
	46	2	7	1 (12)	0
	38	2	8	1 (4)	β_1 for $p=5$
	62	2	9	1 (8)	0
	70	2	10	1 (12)	0
	94	2	11	1 (24)	0
$\mathbf{Za_2}$ [31]	26	2	12	1,3 (8)	0[a]
	38	2	13	1 (8)	0
	58	2	14	1,19 (24)	0
	70	2	15	1 (24)	0
	98	2	16	1 (5)	0
	158	2	17	1 (20)	0
	178	2	18	1 (15)	0
	238	2	19	1 (60)	0
	82	2	20	1,4 (15)	0
	142	2	21	1,49 (60)	0
	62	2	22	1,9 (20)	0
	33	3	23	1,4 (5)	0
$\mathbf{DW_3}$ [11]	45	3	24	1,2,4 (7)	?
	51	3	25	1 (3)	0
	69	3	26	1 (3)	0
	93	3	27	1,4 (15)	0
$\mathbf{F_4}$	52	4	28	Any	?
$\mathbf{Ag_4}$ [3]	84	4	29	1 (4)	0
	124	4	30	1,4 (5)	0
$\mathbf{Za_4}$ [31]	124	4	31	1 (4)	?
	164	4	32	1 (3)	?
	95	5	33	1 (3)	0
$\mathbf{Ag_6}$ [3]	258	6	34	1 (3)	0
$\mathbf{E_6}$	78	6	35	Any	?
$\mathbf{E_7}$	133	7	36	Any	?
$\mathbf{E_8}$	248	8	37	Any	?

[a]Does not vanish for purely dimensional and filtration reasons.

would follow from the Hopkins–Mahowald conjecture for simple groups: Knapp shows in [17] that every rank r compact Lie group represents an element of Adams–Novikov filtration at least r. Since eo_2 and eo_{p-1} only detect homotopy classes up to a certain maximal filtration, depending on p, and since there are only finitely many p-compact groups for every given p and rank r, these two statements imply that only finitely many simple groups can represent nonzero elements. Nishida's nilpotence theorem [21] would then imply that the same holds for arbitrary p-compact groups.

Notation: The symbol $\mathbf{Z}_p$ denotes the p-adic integers. All homology and cohomology theories in this paper are assumed to be reduced, and all spaces to be compactly generated weak Hausdorff.

2. $H\mathbf{Z}/p$-local equivariant spectra

In [7], Bousfield constructs a localization functor $X \to X_E$ for every spectrum E with the property that $X \to X_E$ is the terminal E_*-equivalence out of X. Let L_p denote this functor for $E = H\mathbf{Z}/p$. For connective spectra, this is equivalent to localization with respect to $M(\mathbf{Z}/p)$, the mod-p Moore spectrum, and to p-completion. For a finite spectrum X, $L_pX = X \wedge L_p\mathbf{S}^0$. This and the fact that $L_p(L_p\mathbf{S}^0 \wedge L_p\mathbf{S}^0) \simeq L_p\mathbf{S}^0$ implies that for finite spectra X, Y,

$$L_p(L_pX \wedge L_pY) \simeq L_p(X \wedge Y). \tag{2.1}$$

Let $\mathscr{S}$ be the full subcategory of $H\mathbf{Z}/p$-local spectra. This category has all homotopy limits, homotopy colimits, smash products and function spectra if we compose the usual construction with the functor L_p. (In fact, a homotopy limit of E-local spectra is already E-local.) The smash product is associative up to homotopy, with unit object $L_p\mathbf{S}^0$. When working in $\mathscr{S}$, I will omit any mention of L_p and also write $\mathbf{S}^0$ for the unit of the smash product.

2.1. G-spectra

To construct the transfer map t, we will need to work in a point-set category of equivariant spectra. For our purposes, it is enough to work in the category of so-called naive G-spectra. I will drop the word "naive" since it will make this work appear so puny. Let $G\mathscr{S}$ be the category whose objects are $H\mathbf{Z}/p$-local spectra E, together with a (left) G-action on every space E_n ($n \in \mathbf{Z}$), such that the structure maps $E_n \to \Omega E_{n+1}$ are G-equivariant homeomorphisms. Morphisms are defined as usual. This category again has all homotopy limits and colimits, smash products, and function spectra. The unit is given by $L_p\mathbf{S}^0$ with the trivial G-action. It may be worth pointing out that the G-action on a smash product is the diagonal one, whereas the G-action on $\mathrm{map}(X, Y)$ is given by conjugation.

There are at least two notions of equivariant equivalences in $G\mathscr{S}$, and it is important to distinguish between them.

Definition 5. I will call a G-equivariant map $f : X \to Y$ between G-spectra an *hG-equivalence* if it is a weak equivalence of underlying spectra. It is called a *G-homotopy equivalence* if there is an inverse map up to homotopies through G-equivariant maps.

For a Lie group G, an hG-equivalence f that also induces an equivalence on H-fixed points for every closed subgroup H is sometimes called a weak G-equivalence. By the equivariant Whitehead theorem for spaces with a Lie group action of G (cf. [2,18]), a weak G-equivalence between G-CW complexes is a G-homotopy equivalence; this is false for hG-equivalences in general. For example, the hG-equivalence $EG \to *$ does not have an equivariant inverse.

Define a free G-CW spectrum to be a G-spectrum which is built from cells of the form $\mathbf{S}^n \wedge G_+$.

Lemma 6. *If E is a free G-CW spectrum and $X \to Y$ is an hG-equivalence of G-spectra, then it induces weak equivalences*

$$\mathrm{map}^G(E,X) \xrightarrow{\sim} \mathrm{map}^G(E,Y) \quad \textit{and} \quad E \wedge_G X \xrightarrow{\sim} E \wedge_G Y.$$

Proof. Both equivalences are clear if E is a single cell $\mathbf{S}^n \wedge G_+$ because in that case,

$$\mathrm{map}^G(E,-) = \mathrm{map}(\mathbf{S}^n,-) \quad \text{and} \quad E \wedge_G X = \mathbf{S}^n \wedge X.$$

It follows for finite spectra by induction and the five-lemma, and in general by a direct limit argument. □

For a G-spectrum X, define

$$X_{hG} = EG_+ \wedge_G X = (EG_+ \wedge X)/G \quad \text{and} \quad X^{hG} = \mathrm{map}^G(EG_+, X),$$

where map^G denotes G-equivariant based maps.

The spectrum $\Sigma^\infty EG_+$ is a free G-CW spectrum. Therefore Lemma 6 implies in particular that an hG-equivalence $f: X \to Y$ induces weak equivalences $f^{hH}: X^{hH} \to Y^{hH}$ and $f_{hH}: X_{hH} \to Y_{hH}$ for any subgroup $H < G$. If H is normal in G then these maps are $h(G/H)$-equivalences.

2.2. Duality

For a nonequivariant spectrum X, let $DX =_{\mathrm{def}} \mathrm{map}(X, \mathbf{S}^0)$ be its dual. This spectrum DX will not have good duality properties in general. For instance, there is no guarantee that $D(DX) \simeq X$. We call X *strongly dualizable* if there is a map $\mathbf{S}^0 \xrightarrow{\eta} X \wedge DX$ such that the following diagram commutes up to homotopy:

$$\begin{tikzcd} \mathbf{S} \arrow[r, "\eta"] \arrow[d, "\iota"] & X \wedge DX \arrow[d, "\tau"] \\ \mathrm{map}(X,X) & DX \wedge X \arrow[l, "\nu"'] \end{tikzcd} \tag{2.2}$$

Here τ is the flip involution, ι is adjoint to the identity map $X \to X$, and ν is the map adjoint to

$$X \wedge DX \wedge X \xrightarrow{\mathrm{eval} \wedge \mathrm{id}_X} X.$$

The existence of such a map η is equivalent to ν being a homotopy equivalence. It implies that $D(DX) \simeq X$. Cf. [20].

It turns out that the category $G\mathscr{S}$ contains very few strongly dualizable objects, i.e. objects for which in the above diagram there is an equivariant map η, or equivalently, ν is a G-homotopy equivalence. This is mainly due to the fact that we are considering naive G-spectra. For example, if M is a compact G-manifold, we usually construct a duality morphism η by embedding M equivariantly into some G-representation V, use the Pontryagin–Thom construction to get an equivariant map

$\mathbf{S}^V \to M^v \wedge M_+$, and desuspend by $\mathbf{S}^V$. This last step is impossible in the category of naive G-spectra unless V is a trivial representation, i.e. unless M has a trivial G-action.

If G is a Lie group, and we work in the category of non-naive G-spectra, it is known that a G-CW spectrum is strongly dualizable if and only if it is a wedge summand of a finite G-CW spectrum. It seems plausible that if one succeeded in setting up the "right" category of non-naive G-equivariant spectra for a p-compact group G, all the objects in this work that are nonequivariantly dualizable but do not appear to be strongly dualizable in $G\mathscr{S}$ would actually have a strong dual in that category. From a philosophical point of view, this would be desirable and make some cumbersome technical problems disappear. However, in my opinion, the effort needed for setting up such a category is not warranted by the purposes of the present work.

Let X be a G-spectrum that, as a nonequivariant spectrum, is strongly dualizable. Then the map $v : DX \wedge X \to \mathrm{map}(X, X)$, which is always G-equivariant by naturality, is an hG-equivalence, and η exists but is not necessarily G-equivariant. As should be expected, X will have about half of all the good properties of a strongly dualizable object. For instance, there is a weak equivalence

$$\mathrm{map}^G(A, B \wedge DX) \to \mathrm{map}^G(A \wedge X, B) \tag{2.3}$$

given by

$$f \mapsto A \wedge X \xrightarrow{f \wedge \mathrm{id}_X} B \wedge DX \wedge X \xrightarrow{\mathrm{id} \wedge \mathrm{eval}} B$$

but in general no such map

$$\mathrm{map}^G(A \wedge DX, B) \nrightarrow \mathrm{map}^G(A, X \wedge B).$$

A spectrum or space X is called *p-finite* if $H_*(X; \mathbf{F}_p)$ is totally finite.

Hopkins pointed out to me the proof of the following lemma:

Lemma 7. *For every connective, $H\mathbf{Z}/p$-local, p-finite spectrum X, there is a finite spectrum X' and a p-equivalence $X' \to X$. Similarly, for every simple, $H\mathbf{Z}/p$-local, p-finite space X, there is a finite CW-complex X' and a p-equivalence $X' \to X$.*

Remark. This association is not claimed to be functorial.

Proof. Let X be a spectrum as in the statement, and let $k \in \mathbf{Z}$ be minimal with $H_k(X; \mathbf{F}_p) \neq 0$. We proceed by induction on the size of $H_*(X; \mathbf{F}_p)$.

We will first show that there is a nontrivial map

$$f : \pi_k(X) \to H_k(X) \to H_k(X; \mathbf{F}_p).$$

This would be a simple application of the Hurewicz theorem relative to a Serre class if the class of groups that vanish when tensored with $\mathbf{F}_p$ were actually a Serre class, which it is not.

Since X is connective and $H\mathbf{Z}/p$-local, its $H\mathbf{Z}/p$-nilpotent completion and its $H\mathbf{Z}/p$-localization agree and are equal to X, hence the classical $H\mathbf{Z}/p$-based Adams spectral sequence converges to the homotopy groups of X. Now the Hurewicz map $f : \pi_k(X) \to H_k(X; \mathbf{F}_p)$ has to be nonzero because for $t - s < k$, $\mathrm{Ext}_{A^*}^{s,t}(H^*(X; \mathbf{F}_p), \mathbf{F}_p) = 0$.

Let $\beta : \mathbf{S}^k \to X$ be a map such that $f([\beta]) \neq 0$. Let F be the $H\mathbf{Z}/p$-localization of the homotopy fiber of β. F is p-finite, $H\mathbf{Z}/p$-local, and the size of its $\mathbf{Z}/p$-homology is smaller than that of X, hence by induction, there is a finite spectrum F' and a p-equivalence $F' \to F$. Let X' be the cofiber of $F' \to F \to \mathbf{S}^k$; X' is a finite spectrum and comes with a map $X' \to X$ which is a p-equivalence.

The same argument works for nice enough (e.g., simple) spaces, using the unstable Adams (Bousfield–Kan) spectral sequence. □

Corollary 8. *Let X be a connective, $H\mathbf{Z}/p$-local, p-finite spectrum. Then X has a strong dual in $\mathscr{S}$.*

Proof. By Lemma 7, there is a finite spectrum X' and a p-equivalence $X' \to X$. Hence there is a p-equivalence of $H\mathbf{Z}/p$-local spectra $L_pX' \to L_pX = X$, which therefore is a weak equivalence. It remains to show that $L_p(D(X'))$ is a strong dual of L_pX' for a finite spectrum X'.

We need to show that

$$L_p(\mathrm{map}(X', \mathbf{S}^0)) = L_p\,\mathrm{map}(L_pX', L_p\mathbf{S}^0).$$

Indeed,

$$L_p\,\mathrm{map}(L_pX', L_p\mathbf{S}^0) \simeq \mathrm{map}(X', L_p\mathbf{S}^0) \simeq DX' \wedge L_p\mathbf{S}^0 \simeq L_p(DX').$$

Now $\eta : \mathbf{S}^0 \to X \wedge DX$ induces a duality map

$$L_p\eta : L_p\mathbf{S}^0 \to L_p(X' \wedge DX') \underset{(2.1)}{\xrightarrow{\simeq}} L_pX' \wedge L_pDX' = L_pX' \wedge D(L_pX'),$$

which shows that $L_p(D(X'))$ is a strong dual. □

3. p-compact groups

In this section, I will summarize the definition and some properties of p-compact groups. Most of this is due to Dwyer and Wilkerson [10,12].

Definition 9 (Dwyer and Wilkerson [12]). A *p-compact group* is a triple (X, BX, e) where BX is a $H\mathbf{Z}/p$-local space, X is a p-finite space, and $e : X \to \Omega BX$ is a homotopy equivalence.

We will simply call X a p-compact group when a particular loop structure on X is understood. As noted in the introduction, the $H\mathbf{Z}/p$-localization L_pG of a connected Lie group G gives rise to a p-compact group (L_pG, L_pBG, L_pe) for every prime p. Here $e : G \to \Omega BG$ is the canonical equivalence.

A large class of p-compact groups, called the *nonmodular* groups, can be constructed very easily. Although they constitute fairly uninteresting examples in p-compact group theory, they still give rise to some interesting homotopy classes, therefore I will sketch the construction:

First pick a finite group $W < GL_r(\mathbf{Z}_p)$ (a "Weyl" group for the p-compact group); W acts on $\mathbf{Z}_p^r$ and hence also on $K(\mathbf{Z}_p^r, 2) = L_p(\mathbf{C}P^\infty)^r$. Define a space

$$BG =_{\mathrm{def}} L_p(K(\mathbf{Z}_p^r, 2)_{hW}).$$

We want to determine what restrictions on W we have to make to ensure that BG is a space with polynomial $\mathbf{F}_p$-cohomology. This would imply that the cohomology ring of G is exterior (assume $p > 2$) and hence totally finite.

There is a spectral sequence converging to $H^*(BG;\mathbf{F}_p)$ whose E_2 term is

$$E_2^{r,s} = H^r(BW; H^s(K(\mathbf{Z}_p, 2); \mathbf{F}_p)) = H^r(BW; \mathbf{F}_p[\![t_1, \dots, t_r]\!]).$$

If p does not divide $|W|$ then $E_2^{r,s} = 0$ for $r > 0$, and

$$E_2^{0,s} = \mathbf{F}_p[\![t_1, \dots, t_r]\!]^W = H^s(BG; \mathbf{F}_p).$$

Theorem 10 (Sheppard and Todd [26] and Clark and Ewing [8]). *Let $W < GL_r(\mathbf{F}_p)$ be finite. If $\mathbf{F}_p[t_1, \dots, t_r]^W$ is polynomial then W is a* pseudo-reflection group, *i.e. it is generated by a finite set of finite order elements that fix a hyperplane in $\mathbf{F}_p^r$.*

The converse is true if (but not only if) p does not divide the order of W.

Moreover, in the nonmodular case, every representation of W over $\mathbf{F}_p$ can be lifted to a representation over $\mathbf{Z}_p$.

We can thus construct a p-compact group $G = \Omega BG$ for every pseudo-reflection group defined as a subgroup of $GL_r(\mathbf{Z}_p)$ such that p does not divide the order of W. All such groups are classified [8,26], and Table 1 lists some statistics about them. In that table, all exotic groups of rank bigger than 1 that are given a name are modular.

Definition 11. A morphism $(H, BH, e_H) \to (G, BG, e_G)$ of p-compact groups is just a pointed map $Bf : BH \to BG$. It is a *monomorphism* if its homotopy fiber is p-finite, and an *epimorphism* if its homotopy fiber is BK for a p-compact group (K, BK, e_K).

Two morphisms $BH \to BG$ are called *conjugate* if they are freely homotopic.

In the nonmodular case considered above, BG naturally comes with a map

$$BT =_{\text{def}} K(\mathbf{Z}_p^r, 2) \to L_p(K(\mathbf{Z}_p^r, 2)_{hW}) = BG$$

given by the inclusion of the fiber of the bundle $BG \to BW$. Call a monomorphism of p-compact groups $T \to G$ a *torus* in G if $BT = K(\mathbf{Z}_p^r, 2)$ for some r, and *maximal* if it does not factor through a larger torus. The torus T above is maximal, and one of the main results of [12] is that such maximal tori also exist in the modular case:

Theorem 12 (Dwyer and Wilkerson [12]). (1) *For every connected p-compact group G, there is a maximal torus $BT \to BG$, unique up to conjugacy and automorphisms of the source.*

(2) *The monoid* $\mathrm{map}_{BG}(BT, BT)$ *of endomorphisms of the fibration $BT \to BG$ is homotopically finite with the component group W acting faithfully as a group of pseudo-reflections on $H^2(BT; \mathbf{Z}_p) \cong \mathbf{Z}_p^r$.*

(3) $H^*_{\mathbf{Q}_p}(BG_+) \cong H^*_{\mathbf{Q}_p}(BT_+)^W$, *and $H^*_{\mathbf{Q}_p}(BT_+)$ is a free $H^*_{\mathbf{Q}_p}(BG_+)$-module.*

Here $H^*_{\mathbf{Q}_p}(X) =_{\text{def}} H^*(X; \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. (Note that $H^*(X; \mathbf{Q}_p)$ would be an unreasonably large group; whereas $\mathrm{Hom}(\mathbf{Z}_p, \mathbf{Z}_p) = \mathbf{Z}_p$, we have

$$\mathrm{Hom}(\mathbf{Z}_p, \mathbf{Q}_p) = \mathrm{Hom}(\mathbf{Q}_p, \mathbf{Q}_p) = \mathbf{Q}_p^{\aleph_2}.)$$

Corollary 13. *The p-compact flag variety* $G/T = \mathrm{hofib}(BT \to BG)$ *has*

$$H^*_{\mathbf{Q}_p}(G/T_+) = H^*_{\mathbf{Q}_p}(BT_+)/(H^*_{\mathbf{Q}_p}(BT)^W).$$

Proof. There is an Eilenberg–Moore spectral sequence

$$E_2^{s,t} = \widehat{\mathrm{Tor}}^{-s,t}_{H^*_{\mathbf{Q}_p}(BG_+)}(H^*_{\mathbf{Q}_p}(BT_+), \mathbf{Q}_p) \Rightarrow H^*_{\mathbf{Q}_p}(G/T_+),$$

where $\widehat{\mathrm{Tor}}^s$ is the sth derived functor of the completed tensor product $\hat{\otimes}$. In this spectral sequence, $E_2^{-s,t} = 0$ for $s > 0$ because $H^*_{\mathbf{Q}_p}(BT_+)$ is free, hence flat, over $H^*_{\mathbf{Q}_p}(BG_+)$, and

$$E_2^{0,t} = H^*_{\mathbf{Q}_p}(BT_+)\hat{\otimes}_{H^*_{\mathbf{Q}_p}(BG_+)}\mathbf{Q}_p = H^*_{\mathbf{Q}_p}(BT_+)/(H^*_{\mathbf{Q}_p}(BT)^W). \qquad \square$$

It is important in calculations to know exactly what the degree of the integral edge homomorphism

$$c : H^*(BT_+; \mathbf{Z}_p)/(H^*(BG; \mathbf{Z}_p)) \to H^*(G/T_+; \mathbf{Z}_p)$$

is in the top dimension.

Proposition 14. *Let G be a simply connected p-compact group. Assume that* $H^*(BG; \mathbf{Z}_p)$ *is a polynomial algebra on even-dimensional generators. Then c is an isomorphism.*

Proof. Let $H^*(G; \mathbf{Z}_p) = \mathbf{Z}_p[\![c_1, \ldots, c_n]\!]$. This is a regular local ring with maximal ideal $(p, c_1, \ldots, c_n)$ and Krull dimension $n+1$. $H^*(BT; \mathbf{Z}_p)$ is a finitely generated module over it, and thus the Auslander–Buchsbaum formula [4] implies that $H^*(BT; \mathbf{Z}_p)$ is projective over $H^*(BG; \mathbf{Z}_p)$. Hence, as in the rational case, the Eilenberg–Moore spectral sequence is concentrated on the $s = 0$ line, and the edge homomorphism c is an isomorphism. $\square$

In the definition of a p-compact group (X, BX, e), the data X and e are redundant and probably only classically included to provide some justification for speaking of "a p-compact group X" and not the more accurate "BX". On the other hand, it is always possible to choose a model for the loop space $X = \Omega BX$ such that X is actually a topological group and not just an H-space. A possible construction is the geometric realization of Kan's loop group functor G as described in [15].

This suggests the following alternative definition of a p-compact group:

Equip the category of topological groups with the injective model structure, i.e., weak equivalences and fibrations are the same as for the underlying topological spaces.

Definition 15 (alternative). The category of p-compact groups is the full subcategory of all topological groups whose objects are fibrant and cofibrant in the above sense, and also p-finite, $\mathbf{Z}/p$-local, and such that $\pi_0(G)$ is a finite p-group.

The cofibrancy condition ensures that any morphism $BH \to BG$ is actually induced from a group homomorphism between the chosen models $H \to G$. Up to homotopy, it is even induced by a subgroup inclusion. In fact, we can functorially replace $BH \to BG$ by a cofibration, and Kan's functor G preserves cofibrations. Cofibrations of topological groups are injective. The condition on the group of components is necessary to ensure that BG is still $H\mathbf{Z}/p$-local.

We will therefore work in the category of p-compact groups according to the above alternative definition, and define monomorphisms as actual subgroup inclusions $H \hookrightarrow G$ such that G/H is p-finite.

4. Adjoint representations and self-duality for p-compact groups

4.1. The dualizing spectrum

Although much of Lie theory carries over to the more general setting of p-compact groups, the representation theory, and in particular the adjoint representation, does not seem to have a direct analogue for p-compact groups. We do not know how to construct a vector bundle on a p-compact group BG that plays that role, but we can manufacture something that, in the Lie cases, looks like its Thom spectrum. Following Klein [16], we first construct a dualizing spectrum for G.

Definition 16. For any connected p-compact group G, define

$$S_G = (\Sigma^\infty G_+)^{hG^{\mathrm{op}}}.$$

Note that G acts on $\Sigma^\infty G_+$ by both left and right multiplication. Since the map $g \mapsto g^{-1}$ intertwines the two actions, it makes no difference which one we use for the formation of this homotopy fixed point spectrum. We agree to use the right action, leaving us a left G-action on S_G.

The *adjoint Thom spectrum* of G is the spectrum

$$BG^{\mathfrak{g}} =_{\mathrm{def}} (S_G)_{hG} = EG_+ \wedge_G S_G.$$

Klein [16] showed that this construction for G a (nonlocalized) connected compact Lie group indeed gives rise to the Thom spectrum of the adjoint bundle. It is therefore reasonable to mimick this construction for a p-compact group G. The main point of this section is to show that S_G deserves the name "dualizing spectrum"and is homotopy equivalent to a sphere.

We will need a classical lemma on finite-dimensional Hopf algebras. All cohomology and homology groups are with coefficients in $\mathbf{F}_p$.

Lemma 17. *If G is a topological group such that $H_*(G)$ is totally finite, then $H^*(\Sigma^\infty G_+)$ is a free $H_*(G)$-module on a generator in dimension* $\dim G$.

Proof. Note that $A = H_*(G)$ is a Hopf algebra, and

$$H^*(\Sigma^\infty G_+) \cong A^*$$

by universal coefficients. The dual algebra A^* is a Hopf algebra with antipode c coming from inversion in the group G, and A is a right Hopf module over A^*: the module structure is given by

$$A \otimes A^* \to A, \quad \text{the adjoint map of the coproduct } \psi : A \to A \otimes A$$

and the comodule structure by

$$A \to A \otimes A^*, \quad \text{the adjoint map of the product on A.}$$

Let $P(A)$ denote the $\mathbf{F}_p$-vector space of primitives of A as an A^*-comodule, i.e.

$$P(A) = \{a \in A \mid ax = a\varepsilon(x) \text{ for all } x \in A\},$$

where ε is the augmentation $H_*(G_+) \to H_*(\mathbf{S}^0)$.

Then (cf. [24]), we have a splitting

$$A \xrightarrow{\cong} P(A) \otimes A^*$$

as right A^*-Hopf modules, given by

$$\begin{tikzcd} A \arrow[r, dashed, "\psi"] \arrow[drrrr, dashed] & A \otimes A^* \arrow[r, "\mathrm{id} \otimes \psi"] & A \otimes A^* \otimes A^* \arrow[r, "\mathrm{id} \otimes c \otimes \mathrm{id}"] & A \otimes A^* \otimes A^* \arrow[r, "\mu \otimes \mathrm{id}"] & A \otimes A^* \\ & & & & P(A) \otimes A^* \arrow[u] \end{tikzcd} \tag{4.1}$$

Since A is finite dimensional, it follows that $\dim P(A) = 1$. The assertion of Lemma 17 follows. □

We will show later (Proposition 22) that for G a p-compact group, this map is realizable as a map of spectra.

4.2. Restricted homotopy fixed points

Lemma 18. *Let $H < G$ be a monomorphism of connected p-compact groups. Then for any spectrum X and H-spectrum Y there is a natural hG-equivalence*

$$G_+ \wedge_H \operatorname{map}(X, Y) \xrightarrow{\sim} \operatorname{map}(X, G_+ \wedge_H Y).$$

Proof. The map is adjoint to

$$(G_+ \wedge_H \operatorname{map}(X, Y)) \wedge X \xrightarrow{\sim} G_+ \wedge_H (\operatorname{map}(X, Y) \wedge X) \xrightarrow{G_+ \wedge_H \mathrm{ev}} G_+ \wedge_H Y$$

and thus clearly natural and G-equivariant. We now claim that G has the structure of a finite free H-CW complex. The following argument was suggested by the referee. By Lemma 7 let F be a finite CW-complex with a p-equivalence $F \to G/H$, and let E be the pullback of $F \to G/H \leftarrow G$. Then $E \to G$ is a p-equivalence.

Let $\mathbf{D}^d \xrightarrow{\alpha} F$ be a cell and C_α the pullback of α along $E \to F$. Then $C_\alpha = H \times \mathbf{D}^d$ as H-spaces, and the collection of C_α constitute a finite H-CW structure on E and thus induce an $H\mathbf{Z}/p$-local H-CW structure on G.

Now for any free finite H-CW complex Z, we have that

$$Z_+ \wedge_H \operatorname{map}(X, Y) \to \operatorname{map}(X, Z_+ \wedge_H Y)$$

is an equivalence. By induction on the cells, it is enough to check it for a free cell $Z = \mathbf{S}^n \wedge H_+$, for which it is obvious because $\mathbf{S}^n$ is nonequivariantly small. □

Lemma 19. *For a connected sub-p-compact group $H < G$ in a connected p-compact group G, there is an hG-equivalence*

$$G_+ \wedge_H S_H \xrightarrow{\sim} (\Sigma^\infty G_+)^{hH^{\mathrm{op}}}.$$

Proof. First note that as $(G \times H^{\mathrm{op}})$-spectra,

$$\Sigma^\infty G_+ \simeq G_+ \wedge_H \Sigma^\infty H_+,$$

where on the right-hand side, H acts on the right factor from the right and G acts on the left factor from the left.

We therefore have a map

$$G_+ \wedge_H \operatorname{map}(EH_+, \Sigma^\infty H_+) \to \operatorname{map}(EH_+, G_+ \wedge_H \Sigma^\infty H_+) \simeq \operatorname{map}(EH_+, \Sigma^\infty G_+).$$

By Lemma 18 and naturality, this map is an $h(G \times H^{\mathrm{op}})$-equivalence, hence passing to H^{op}-homotopy fixed points we get that

$$\begin{aligned} G_+ \wedge_H S_H &= G_+ \wedge_H \operatorname{map}^{H^{\mathrm{op}}}(EH_+, \Sigma^\infty H_+) \\ &\to \operatorname{map}^{H^{\mathrm{op}}}(EH_+, G_+ \wedge_H \Sigma^\infty H_+) = (\Sigma^\infty G_+)^{hH^{\mathrm{op}}}. \end{aligned}$$

is an hG-equivalence □

If $X \in (G \times G^{\mathrm{op}})\mathscr{S}$ and $Y \in H^{\mathrm{op}}\mathscr{S}$, we have G-equivariant homotopy equivalences (given by shearing maps)

$$G_+ \wedge_H X \simeq G/H_+ \wedge X$$

and

$$\operatorname{map}^{H^{\mathrm{op}}}(Y \wedge G_+, X) \simeq \operatorname{map}(Y \wedge_H G_+, X).$$

In particular, if $Y \in (G \times G^{\mathrm{op}})\mathscr{S}$, we have

$$\operatorname{map}^{H^{\mathrm{op}}}(Y \wedge G_+, X) \simeq \operatorname{map}(Y \wedge G/H_+, X). \tag{4.2}$$

Lemma 20. *Let $H < G$ be as above. Then there is an hG-equivalence*

$$(DG_+)^{hH^{\mathrm{op}}} \xrightarrow{\simeq} D(G/H_+),$$

natural on subgroups of G.

Proof. The map is the following composite of hG-equivalences, all of which are natural:

$$\begin{aligned} (DG_+)^{hH^{\mathrm{op}}} &= \operatorname{map}^H(EH_+, DG_+) \\ &\simeq \operatorname{map}^H(EG_+, DG_+) \\ &\to \operatorname{map}^H(EG_+ \wedge G_+, \mathbf{S}^0) \\ &\xrightarrow{f} \operatorname{map}(EG_+ \wedge G/H_+, \mathbf{S}^0) \\ &\xrightarrow{g} D(G/H_+). \end{aligned}$$

For the first homotopy equivalence, we use that EG is a valid model for EH. f is a G-equivariant homotopy equivalence by (4.2). Since EG_+ has the usual right action and a trivial left action, the map $\mathbf{S}^0 \to EG_+$ is a left G-homotopy equivalence, and hence so is g. □

4.3. Absolute self-duality

Denote by G_c the space G with the conjugation action of G. For G a Lie group, S_G can be identified with the suspension spectrum of the one-point compactification of a neighborhood of the identity in G; this identification is G-equivariant if we equip G with the conjugation action. Thus we obtain an equivariant "logarithm" $\Sigma^\infty(G_c)_+ \to S_G$. The following lemma shows that a similar map also exists for p-compact groups, at least up to an hG-equivalence. Furthermore, it has an equivariant splitting, a fact that is not obvious even for Lie groups.

Lemma 21. *For every connected p-compact group G, there is a G-spectrum $Q(G)$, a natural hG-equivalence $Q(G) \to \Sigma^\infty(G_c)_+$, and a G-equivariant retraction $Q(G) \to S_G$.*

Remark. An equivariant retraction $X \to Y$ means two equivariant maps

$$Y \to X \to Y$$

such that the composite is an hG-equivalence.

Proof. The auxiliary spectrum $Q(G)$ is defined as

$$Q(G) = \mathrm{map}^{G^{\mathrm{op}}}(EG_+, \Sigma^\infty G_+ \wedge DG_+).$$

Consider $\Sigma^\infty G_+$ as a $(G \times G^{\mathrm{op}})$-spectrum by left and right multiplication. Then the diagonal map

$$\Sigma^\infty G_+ \to \Sigma^\infty G_+ \wedge \Sigma^\infty G_+$$

is $(G \times G^{\mathrm{op}})$-equivariant and has an equivariant adjoint

$$\Sigma^\infty G_+ \wedge DG_+ \to \Sigma^\infty G_+. \tag{4.3}$$

Similarly, the $(G \times G^{\mathrm{op}})$-equivariant projection map to the first factor

$$\Sigma^\infty G_+ \wedge \Sigma^\infty G_+ \to \Sigma^\infty G_+$$

has an equivariant adjoint

$$\Sigma^\infty G_+ \to DG_+ \wedge \Sigma^\infty G_+.$$

The composite

$$\Sigma^\infty G_+ \to \Sigma^\infty G_+ \wedge DG_+ \to \Sigma^\infty G_+ \tag{4.4}$$

is a weak equivalence.

Taking homotopy fixed points with respect to $G^{\mathrm{op}} = 1 \times G^{\mathrm{op}} \subseteq G \times G^{\mathrm{op}}$ on the left-hand side of (4.3) yields

$$\begin{aligned} Q(G) = \mathrm{map}^{G^{\mathrm{op}}}(EG_+, \Sigma^\infty G_+ \wedge DG_+) &\underset{(2.3)}{\xrightarrow{\sim}} \mathrm{map}^{G^{\mathrm{op}}}(EG_+ \wedge G_+, \Sigma^\infty G_+) \\ &\xrightarrow{\sim} \mathrm{map}(EG_+, \Sigma^\infty G_+) \\ &\xrightarrow{\sim} \mathrm{map}(\mathbf{S}^0, \Sigma^\infty G_+) \simeq \Sigma^\infty G_+. \end{aligned} \tag{4.5}$$

As in Lemma 20, the map induced by $\mathbf{S}^0 \to EG_+$ is indeed a G-homotopy equivalence because the left G-action on EG_+ is trivial. In fact, all maps but the first one are G-homotopy equivalences.

We have to check that the G-action on $Q(G)$ corresponds to the conjugate action on $\Sigma^\infty G_+$. The action of G on $M = \mathrm{map}^{G^{\mathrm{op}}}(EG_+ \wedge G_+, \Sigma^\infty G_+)$ is given by

$$(g.f)(x \wedge \gamma) = gf(x \wedge g^{-1}\gamma) \quad (g \in G,\ f \in M,\ x \in EG_+,\ \gamma \in G_+).$$

The induced action of G on $\mathrm{map}(EG_+, \Sigma^\infty G_+)$ is

$$(g.f)(x) = gf(xg)g^{-1} \quad (g \in G,\ f \in \mathrm{map}(EG_+, \Sigma^\infty G_+),\ x \in EG_+)$$

since

$$\begin{aligned} \mathrm{map}^{G^{\mathrm{op}}}(EG_+ \wedge G_+, \Sigma^\infty G_+) &\to \mathrm{map}(EG_+, \Sigma^\infty G_+), \\ f &\mapsto x \mapsto f(x,1), \\ g.f &\mapsto x \mapsto gf(x, g^{-1}) = gf(xg,1)g^{-1}. \end{aligned}$$

The restricted G-action on $\mathrm{map}(\mathbf{S}^0, \Sigma^\infty G_+)$ becomes

$$(g.f)(x) = gf(x)g^{-1}$$

since $\mathbf{S}^0$ has the trivial G-action, and the G-action on $\Sigma^\infty G_+$ is indeed by conjugation.

Applying $(-)^{hG^{\mathrm{op}}}$ to (4.4) yields the desired retraction $Q(G) \to S_G$. This retraction is just the canonical map $(\Sigma^\infty G_+)^{hG} \to \Sigma^\infty G_+$ from the homotopy fixed points into the full space. □

Proposition 22. *Regard the G-spectrum S_G as a $(G \times G^{\mathrm{op}})$-spectrum with trivial G^{op}-action. Then there is an $h(G \times G^{\mathrm{op}})$-equivalence*

$$S_G \wedge DG_+ \xrightarrow{\sim} \Sigma^\infty G_+.$$

On G^{op}-homotopy fixed points, these maps make the following diagram commute:

$$\begin{tikzcd} S_G = (\Sigma^\infty G_+)^{hG^{\mathrm{op}}} \arrow[drr, equal] & (S_G \wedge DG_+)^{hG^{\mathrm{op}}} \arrow[l, "\sim"'] & S_G \wedge (DG_+)^{hG^{\mathrm{op}}} \arrow[l, "\sim"'] \arrow[d, "\sim"', "\text{Lemma 20}"] \\ & & S_G \wedge \mathbf{S}^0 \end{tikzcd}$$

Corollary 23. *S_G is homotopy equivalent to a $H\mathbf{Z}/p$-local sphere of dimension d.*

Proof of the corollary. As a homotopy fixed point spectrum, S_G is still $H\mathbf{Z}/p$-local. The proposition implies that

$$H_*(G; \mathbf{F}_p) \cong H_*(S_G; \mathbf{F}_p) \otimes H^{-*}(G; \mathbf{F}_p).$$

By Lemma 17, $H^*(G; \mathbf{F}_p)$ is free of rank one over $H_*(G; \mathbf{F}_p)$ on a generator in dimension d, and $H_*(S_G; \mathbf{F}_p) \cong H_*(\mathbf{S}^d; \mathbf{F}_p)$. The proof of Lemma 7 produces a p-equivalence $\mathbf{S}^d \to S_G$ in that case. □

Proof of the proposition. We will have to deal with spectra with three G-actions, and for ease of notation, for a $(G \times G^{\mathrm{op}})$-spectrum X, I will denote by ${}^aX^b_c$ the spectrum X with the left action a and the two right actions b and c, where a, b, c, are one of the following:

- '$\circ$' denotes a trivial action.
- 'l' denotes the action from the left—if this symbol appears on the right then G acts by inverses from the left.
- 'r' denotes the action from the right—if this symbol appears on the left then G acts by inverses from the right.

The main ingredient is a shearing map

$$ {}^l(\Sigma^\infty G_+)^\circ_r \wedge {}^l(DG_+)^r_\circ \xrightarrow{\mathrm{sh}} {}^l(\Sigma^\infty G_+)^\circ_r \wedge {}^\circ (DG_+)^l_r, \tag{4.6}$$

which is adjoint to

$$ \begin{aligned} {}^l(\Sigma^\infty G_+)^\circ_r \wedge {}^\circ (\Sigma^\infty G_+)^l_r &\xrightarrow{\mathrm{sh}'} {}^l(\Sigma^\infty G_+)^\circ_r \wedge {}^l(\Sigma^\infty G_+)^r_\circ \\ g \wedge h \ &\mapsto \ g \wedge gh^{-1}. \end{aligned} $$

The map sh' is clearly a homotopy equivalence, and it is easy to check that it is $(G \times G^{\mathrm{op}} \times G^{\mathrm{op}})$-equivariant as claimed. One checks that the adjoint map sh is also a homotopy equivalence whose inverse is the adjoint of the inverse of sh'.

By passing to homotopy fixed points with respect to the ${}^\circ\square^\circ_\bullet$ action of G^{op} in (4.6), we obtain a $(G \times G^{\mathrm{op}})$-equivariant homotopy equivalence

$$\begin{tikzcd} ({}^l(\Sigma^\infty G_+)^\circ_r \wedge {}^l(DG_+)^r_\circ)^{hG^{\mathrm{op}}} \arrow[r] & ({}^l(\Sigma^\infty G_+)^\circ_r \wedge {}^\circ (DG_+)^l_r)^{hG^{\mathrm{op}}} =_{\mathrm{def}} Q'(G) \arrow[d, "\sim"'] \\ {}^l(S_G)^\circ \wedge {}^l(DG_+)^r \arrow[u, "\sim"] & {}^l(\Sigma^\infty G_+)^r \end{tikzcd}$$

The underlying spectrum of $Q'(G)$ is the spectrum $Q(G)$ of Lemma 21. It is easy to see that with the remaining operations, the map $Q'(G) \to \Sigma^\infty G_+$, described in (4.5), is $(G \times G^{\mathrm{op}})$-equivariant.

For the assertion about G^{op}-homotopy fixed points, observe that by changing the order of taking G^{op}-homotopy fixed points, we have a large commutative diagram

$$\begin{tikzcd} {}^lS_G = {}^lG^{hG^{\mathrm{op}}} \arrow[rr, bend left, equal] & {}^lS_G \wedge {}^\circ(DG)^{hG^{\mathrm{op}}} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\sim"'] & {}^lS_G \arrow[d, equal] \\ \left({}^lG^\circ_r \wedge {}^\circ(DG)^l_r\right)^{h^\circ\square^\bullet_\bullet} \arrow[u, "\sim"] & \left({}^lG^\circ_r \wedge {}^\circ(DG)^{hG^{\mathrm{op}}}_\circ\right)^{h^\circ\square_\bullet} \arrow[l, "\sim"'] \arrow[r, "\sim"] & {}^lS_G \arrow[d, equal] \\ \left({}^lG^\circ_r \wedge {}^l(DG)^r_\circ\right)^{h^\circ\square^\bullet_\bullet} \arrow[u, "\sim", "\mathrm{sh}"'] & \left({}^lG_r \wedge {}^l(DG)^{hG^{\mathrm{op}}}_\circ\right)^{h^\circ\square_\bullet} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u, "\sim", "\mathrm{sh}"'] & {}^lS_G \\ \left({}^l(S_G)^\circ \wedge {}^l(DG)^r\right)^{h^\circ\square^\bullet} \arrow[u, "\sim"] & {}^lS_G \wedge {}^l(DG)^{hG^{\mathrm{op}}} \arrow[l, "\sim"'] \arrow[u, "\sim"] & \end{tikzcd}$$

For space and readability reasons, the disjoint basepoint for $\Sigma^\infty G$ and DG have been omitted as well as the suspension functor Σ^∞ for G.

The important, if trivial, observation is that the shear map becomes homotopic to the identity when passing to °□:-homotopy fixed points on the DG_+ factor. The diagram claimed to be commutative in the proposition is the "boundary" of the diagram above. □

4.4. Relative self-duality

We will now prove the following, stronger version of Theorem 3 from the introduction:

Corollary 24. *For any inclusion $H < G$ of connected p-compact groups, here is a zigzag of hG-equivalences*

$$G_+ \wedge_H S_H \overset{\sim}{\longleftrightarrow} D(G/H_+) \wedge S_G.$$

This zigzag is natural in the following sense: for any chain of p-compact groups $K < H < G$, the following diagram commutes:

$$\begin{tikzcd}
(\Sigma^\infty G_+)^{hH^{\mathrm{op}}} \arrow[d, "\mathrm{res}"] & G_+ \wedge_H S_H \arrow[l, "\sim"'] \arrow[r, dotted, leftrightarrow, "\sim"] & D(G/H_+) \wedge S_G \arrow[d, "D(\mathrm{proj}) \wedge \mathrm{id}"] \\
(\Sigma^\infty G_+)^{hK^{\mathrm{op}}} & G_+ \wedge_K S_K \arrow[l, "\sim"'] \arrow[r, dotted, leftrightarrow, "\sim"] & D(G/K_+) \wedge S_G
\end{tikzcd}$$

Proof. From Proposition 22, we have an $h(G \times H^{\mathrm{op}})$-equivalence

$$DG_+ \wedge S_G \to \Sigma^\infty G_+.$$

Applying H^{op}-homotopy fixed points turns $h(G \times H^{\mathrm{op}})$-equivalences into hG-equivalences, and since the right actions on S_G and S_H are trivial, we obtain

$$\begin{tikzcd}
(DG_+ \wedge S_G)^{hH^{\mathrm{op}}} \arrow[d, "\sim"] & DG_+^{hH^{\mathrm{op}}} \wedge S_G \arrow[l, "\sim"'] \arrow[r, "\sim"] & D(G/H_+) \wedge S_G \\
(\Sigma^\infty G_+)^{hH^{\mathrm{op}}} & G_+ \wedge_H S_H \arrow[l, "\text{Lemma 19}"', "\sim"]
\end{tikzcd}$$

For naturality, consider the following diagram:

$$\begin{tikzcd}
D(G/H_+) \wedge S_G \arrow[d] & DG_+^{hH^{\mathrm{op}}} \wedge S_G \arrow[l] \arrow[r] \arrow[d] & (DG_+ \wedge S_G)^{hH^{\mathrm{op}}} \arrow[r] \arrow[d] & (\Sigma^\infty G_+)^{hH^{\mathrm{op}}} \arrow[d] \\
D(G/K_+) \wedge S_G & DG_+^{hK^{\mathrm{op}}} \wedge S_G \arrow[l] \arrow[r] & (DG_+ \wedge S_G)^{hK^{\mathrm{op}}} \arrow[r] & (\Sigma^\infty G_+)^{hK^{\mathrm{op}}}
\end{tikzcd}$$

The left-hand square commutes by Lemma 20, the other two for trivial reasons. □

4.5. Definition of the transfer

Proof of Theorem 2. By Corollary 23, S_G is an $H\mathbf{Z}/p$-local sphere. We obtain a (nonequivariant) map

$$\bar{t} : S_G = (\Sigma^\infty G_+)^{hG^{\mathrm{op}}} \to (\Sigma^\infty G_+)^{hH^{\mathrm{op}}} \underset{f}{\overset{\sim}{\to}} G_+ \wedge_H S_H$$

coming from restricting from G- to H-homotopy fixed points. Here f is the nonequivariant homotopy inverse of the hG-equivalence given by Lemma 19.

By Lemma 6, there is also a G-equivariant map

$$\tilde{t}: EG_+ \wedge S_G \to G_+ \wedge_H S_H$$

such that the composite

$$S_G \to EG_+ \wedge S_G \to G_+ \wedge_H S_H$$

is homotopic to $\tilde{t}$, and $\tilde{t}$ is unique up to homotopy with this property.

To finish the proof of Theorem 2, we need to show that

$$\tilde{t}_*: H_d(S_G; \mathbf{F}_p) \to H_d(G_+ \wedge_H S_H; \mathbf{F}_p)$$

is an isomorphism for $d = \dim G$. This now follows easily from Corollary 24: By construction, the map $\tilde{t}$ is the composite

$$S_G \to D(G/H_+) \wedge S_G \xrightarrow{\sim} G_+ \wedge_H S_H.$$

Since the first map is an isomorphism in $H_d(-; \mathbf{F}_p)$, so is the composite. $\square$

The first part of Theorem 4 claims that $H^*(BG^{\mathfrak{g}}; \mathbf{F}_p)$ is a Thom module over $H^*(BG_+; \mathbf{F}_p)$. This follows from the spectral sequence

$$E_2 = H^*(BG_+; H^*(S_G; \mathbf{Z}_p)) \Rightarrow H^*(EG_+ \wedge_G S_G; \mathbf{F}_p) = H^*(BG^{\mathfrak{g}}; \mathbf{Z}_p). \qquad \square$$

Definition 25. For a monomorphism $H < G$ of connected p-compact groups, the transfer map $t_{G,H}$ is given by applying G-homotopy orbits to the G-equivariant map $\tilde{t}$.

The domain of $\tilde{t}$ is $EG_+ \wedge_G (EG_+ \wedge S_G)$, which by Lemma 6 is homotopy equivalent to $BG^{\mathfrak{g}}$. For the functoriality, it is sufficient to note that the following diagram of G-equivariant maps commutes:

$$\begin{tikzcd}
S_G \arrow[r] \arrow[dr] & (\Sigma^\infty G_+)^{hH^{\mathrm{op}}} & G_+ \wedge_H S_H \arrow[l] \arrow[d] \\
 & (\Sigma^\infty G_+)^{hK^{\mathrm{op}}} & G_+ \wedge_H (\Sigma^\infty H_+)^{hK^{\mathrm{op}}} \\
 & & G_+ \wedge_K S_K \arrow[ul] \arrow[u]
\end{tikzcd}$$

That is a less than remarkable statement since no two maps are composable. But all of the maps going left or up or both are hG-equivalences, and the diagram stays (nonequivariantly) homotopy commutative if we invert them.

By its definition, the commutativity of

$$\begin{tikzcd}
S_G \arrow[r] \arrow[d] & G_+ \wedge_H S_H \arrow[d] \\
BG^{\mathfrak{g}} \arrow[r] & BH^{\mathfrak{h}}
\end{tikzcd}$$

as claimed in Theorem 4 is immediate.

The next section will be devoted to identifying the transfer map on the category of $H\mathbf{Z}/p$-localizations of compact Lie groups and monomorphisms.

5. Identification of the transfer map

Using the construction of t for $\{1\} < G$, we have a commutative diagram coming from the natural transformation $\mathrm{id} \to (-)_{hG}$:

$$\begin{tikzcd} S_G \arrow[r] \arrow[d] & \Sigma^\infty G_+ \arrow[d] & \\ BG^{\mathfrak{g}} \arrow[r, "t"] & B\{1\}_+ \arrow[r, equal] & \mathbf{S}^0 \end{tikzcd} \tag{5.1}$$

We will now identify t with the transfer map from the introduction in the case where G is of Lie type.

First note that in (5.1), the composite map $\mathbf{S}^d \to \mathbf{S}^0$ is indeed the same as the Pontryagin–Thom construction on G if G is a Lie group: The Pontryagin–Thom construction on G is given by the composition of maps

$$\mathbf{S}^0 \to DG_+ \simeq \mathbf{S}^{-d} \wedge G_+ \to \mathbf{S}^{-d},$$

where the first map is a desuspension of the map from the embedding sphere to the Thom space of the normal bundle of G, which is DG; by Proposition 22 or since the tangent bundle of G is trivial, this is equivalent to a desuspension of $\Sigma^\infty G_+$; and the second map is the projection of G_+ to $\mathbf{S}^0$, the map classifying the (trivial) stable normal bundle of G.

Using the G-equivariant isomorphism from Proposition 22, we have

$$\begin{tikzcd} S_G \arrow[r] \arrow[dr] & \Sigma^\infty G_+ \arrow[r, "(-)_{hG}"] \arrow[d, dotted, leftrightarrow, "\sim"] & \mathbf{S}^0 \\ & S_G \wedge DG_+ & \end{tikzcd}$$

The bottom composition is the Pontryagin–Thom construction, the upper one the map from (5.1).

5.1. An alternative construction of the transfer map

To show that t agrees with the Umkehr map not only on the bottom cell, we will compare its definition to another, equally general construction, reminiscent of Dwyer's construction of the Becker–Gottlieb transfer in [9]. This will be equivalent to the classical construction of the Umkehr map in the Lie case.

Let $H < G$ be p-compact groups. The quotient $\Sigma^\infty G/H_+$ is dualizable, and the projection $D(G/H_+) \to \mathbf{S}^0$ onto the top cell is equivariant and has a section α; however, α is not G-equivariant unless $H = G$. But we do get an equivariant map if we "free up" the G-action on $\mathbf{S}^0$: consider the following diagram of G-equivariant maps:

$$\begin{tikzcd} \Sigma^\infty EG_+ \arrow[r, "\sim"] & \mathbf{S}^0 \arrow[r] & \mathrm{map}(\Sigma^\infty G/H_+, \Sigma^\infty G/H_+) & \\ & & D(G/H)_+ \wedge \Sigma^\infty G/H_+ \arrow[u, "\eta", "\sim"'] \arrow[r, "\mathrm{id}\wedge\pi"] & D(G/H)_+ \wedge \mathbf{S}^0. \end{tikzcd}$$

The map π is the projection $\Sigma^\infty G/H_+ = \Sigma^\infty G/H \vee \mathbf{S}^0 \to \mathbf{S}^0$, and η is an hG-equivalence. Therefore, by Lemma 6, there exists an equivariant lifting

$$\Sigma^\infty EG_+ \to D(G/H_+), \tag{5.2}$$

which is nonequivariantly homotopic to the map

$$\Sigma^\infty EG_+ \to \mathbf{S}^0 \xrightarrow{\alpha} D(G/H_+).$$

In the case where H and G are (localizations of) Lie groups, the homotopy orbit space of $D(G/H_+)$ under this G-action is the (localization of the) Thom space of v, the normal bundle along the fibers of $BH \to BG$. This follows from the observation that $BH = EG \times_G G/H$, and that stably, the normal bundle along the fiber is the fiberwise Spanier–Whitehead dual, i.e. $v = EG \times_G D(G/H)$. Hence its Thom spectrum is $EG_+ \wedge_G D(G/H_+)$, as claimed.

By passage to G-homotopy orbits in (5.2), we therefore obtain a map

$$\Sigma^\infty BG_+ \simeq EG_+ \wedge_G \Sigma^\infty EG_+ \to BH^{-\mathfrak{g}/\mathfrak{h}},$$

where $BH^{-\mathfrak{g}/\mathfrak{h}}$ denotes the Thom spectrum of the virtual inverse of the adjoint bundle of G, pulled back to BH, modulo the adjoint bundle of H. This is the Lie theoretic model of the normal bundle along the fibers.

Returning to the case of a general p-compact group, we now introduce a “twisting” by smashing source and target of the map with S_G:

$$EG_+ \wedge S_G \to D(G/H_+) \wedge S_G \underset{\text{Cor. 24}}{\overset{\sim}{\longleftrightarrow}} G_+ \wedge_H S_H.$$

By Lemma 6 and since $EG_+ \wedge S_G$ is a free G-spectrum, we obtain a G-map (unique up to homotopy)

$$\tilde{t}' : EG_+ \wedge S_G \to G_+ \wedge_H S_H$$

and passing to G-homotopy orbits, we obtain:

$$t' : BG^{\mathfrak{g}} = EG_+ \wedge_G S_G \to EH_+ \wedge_H S_H = BH^{\mathfrak{h}}.$$

Lemma 26. *Let $H \hookrightarrow G$ be a monomorphism of connected p-compact groups. Then*

$$\tilde{t} \simeq \tilde{t}' : EG_+ \wedge S_G \to G_+ \wedge_H S_H.$$

Proof. We have to show that the following G-equivariant diagram commutes:

$$\begin{tikzcd}
EG_+ \wedge S_G \arrow[r, "(5.2)"] \arrow[d, "\tilde{t}"] & D(G/H_+) \wedge S_G \arrow[d, dotted, leftrightarrow, "\sim"] \\
G_+ \wedge_H S_H \arrow[r, dotted, leftrightarrow, "\sim"] & (DG_+ \wedge S_G)^{hH^{\mathrm{op}}}
\end{tikzcd}$$

Since $EG_+ \wedge S_G$ is a free G-spectrum, Lemma 6 asserts there is a G-equivariant map going diagonally

$$EG_+ \wedge S_G \to (DG_+ \wedge S_G)^{hH^{\mathrm{op}}}$$

and making the upper right triangle of the diagram commute. The commutativity of the lower left triangle then follows from the observation that in the commutative diagram

$$\begin{tikzcd}
S_G \arrow[d] & (DG_+ \wedge S_G)^{hG^{\mathrm{op}}} \arrow[l, "\sim"'] \arrow[r] \arrow[d] & (DG_+ \wedge S_G)^{hH^{\mathrm{op}}}, \arrow[d] \\
S_G \arrow[r, equal] & (\Sigma^\infty G_+)^{hG^{\mathrm{op}}} \arrow[r] & (\Sigma^\infty G_+)^{hH^{\mathrm{op}}}
\end{tikzcd}$$

the left-hand map is the identity by Proposition 22. □

Conclusion of the proof of Theorem 4. The previous lemma implies the third part of the theorem (namely, that $t \circ L_p \simeq L_p \circ (-)_!$ on compact Lie groups and monomorphisms). Indeed, by applying G-homotopy orbits to the diagram in Lemma 26, the map induced by $\tilde{t}'$ is homotopic to t, and the preceding discussion shows that the former map is the classical Umkehr map in the Lie case. □

6. Computational methods

In this section, I will describe a general method for computing the homotopy class represented by a p-compact group by constructing a representing cycle in the Adams spectral sequence for a complex oriented cohomology theory E. Let G be a simply connected d-dimensional p-compact group of rank r with maximal torus T. We want to identify the maps the following composition induces on the E_2-term for the E-cohomology ASS:

$$\mathbf{S}^d \to BG^{\mathfrak{g}} \to BT^{\mathfrak{t}} \to \mathbf{S}^0.$$

6.1. The $\mathbf{S}^1$-transfer

The right-hand map is a suspension of the r-fold smash product of the $\mathbf{S}^1$-transfer map

$$\tau : \mathbf{C}P^\infty_+ \to \mathbf{S}^{-1}.$$

It is well-known that the homotopy fiber of this transfer map is the spectrum $\mathbf{C}P^\infty_{-1}$, the Thom spectrum of the inverse of the universal line bundle on $\mathbf{C}P^\infty$, the fiber inclusion $\mathbf{C}P^\infty_{-1} \to \mathbf{C}P^\infty_+$ being the obvious projection map onto the quotient of $\mathbf{C}P^\infty_{-1}$ by the (-1)-skeleton.

For a complex oriented cohomology theory E and a finite spectrum X, there is an Adams–Novikov spectral sequence

$$E_2 = \operatorname{Ext}(E^{-*}(X)) \Rightarrow [X, L_E\mathbf{S}],$$

where, for an (E_*, E_*E)-comodule A, $\operatorname{Ext}(A)$ is a shorthand for $\operatorname{Ext}_{E_*E}(E_*, A)$, and $E^{-*}(X)$ becomes an (E_*, E_*E)-comodule by identifying $E^{-*}(X)$ with $E_*(DX)$.

We will now first restrict to the case of finite dimensional projective spaces and study the map $\mathbf{C}P^m_+ \to \mathbf{S}^{-1}$ as a map of E_2-terms of this ANSS

$$\begin{array}{ccc} \operatorname{Ext}(E^{-*}(\mathbf{S}^{-1})) & \Longrightarrow & \pi_*(L_E\mathbf{S}^1) \\ \downarrow & & \downarrow \\ \operatorname{Ext}(E^{-*}(\mathbf{C}P^m_+)) & \Longrightarrow & [\mathbf{C}P^m_+, L_E\mathbf{S}]. \end{array}$$

As usual, let $MU_* = \mathbf{Z}[a_i]$ and $MU_*MU = MU_*[b_i]$. Fix an orientation $z \in E^2(\mathbf{C}P^\infty)$ of E. Denote by $a_i \in E_*$, $b_i \in E_*E$ the image of the classes of the same name under the map $MU \to E$ induced by z.

The E_*E-comodule structure on $E^*(\mathbf{C}P^\infty_+)$ is given by

$$\psi(z) = \sum_{i=0}^{\infty} b_i z^{i+1} \in (E \wedge E)^*(\mathbf{C}P^m_+),$$

which is proved in the universal case in [1, Proposition II.9.4].

Let L be the universal line bundle on $\mathbf{C}P^k$ (any k), and let $\mathbf{C}P_n^m$ denote the Thom space of the virtual bundle nL over $\mathbf{C}P^{m-n+1}$. For nonnegative n, this is the truncated projective space $\mathbf{C}P^m/\mathbf{C}P^{n-1}$ with cells in dimensions $2n, 2m+2, \ldots, 2m$. The E_*E-comodule $E^*(\mathbf{C}P_n^m)$ is still a free E_*-module, generated by $\{z^n, z^{n+1}, \ldots, z^m\}$, and the above formula for ψ is correct when interpreted as $\eta_R(z^i) = \eta_R(z)^i$ and truncated at $i = m+1$.

By a change of rings isomorphism, the Adams–Novikov spectral sequence for $X = \mathbf{C}P_n^m$ is isomorphic to the one associated to the Hopf algebroid

$$(A_m(n), \Gamma_m(n)) = (E^{-*}(\mathbf{C}P_n^m), (E \wedge E)^{-*}(\mathbf{C}P_n^m)). \tag{6.1}$$

Under this isomorphism,

$$\eta_R^{(A_m(n),\Gamma_m(n))}(z) = \psi(z).$$

Remark. In the language of algebraic geometry, a complex oriented theory E gives rise to a formal group $G_E = \operatorname{Spf} E^*(\mathbf{C}P^\infty)$ (cf. for example [29]). The module $E^*(\mathbf{C}P_n^m)$ is then the sheaf of functions on G_E which vanish to the nth order at the identity of G_E, modulo functions which vanish to the $(m+1)$st order.

Thus $A_m(n)$ represents the following functor:

$$R \mapsto \left\{ (\alpha, f) \;\middle|\; \begin{array}{l} E_* \xrightarrow{\alpha} R, \\ f \text{ is a function modulo degree } m+1 \text{ on the} \\ \text{formal group on } R \text{ given by the image} \\ \text{of the universal formal group under} \\ MU_* \to E_* \to R \text{ such that } f \text{ vanishes to the} \\ n\text{th order at the identity.} \end{array} \right\}.$$

Similarly, $\Gamma_m(n)$ represents isomorphisms of such data. Hence, for $E = MU$, $(A_m(n), \Gamma_m(n))$ classifies formal groups with an $m+1$-truncated function on it that vanishes to the nth order at the identity.

Assembling all spectral sequences for varying $m \geqslant n$, we obtain towers

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\operatorname{Ext}_{\Gamma_{m+1}}(A_{m+1}(n), A_{m+1}(n)) \arrow[r, Rightarrow] \arrow[d] & {[\mathbf{C}P_n^{m+1}, L_E\mathbf{S}]} \arrow[d] \\
\operatorname{Ext}_{\Gamma_m}(A_m(n), A_m(n)) \arrow[r, Rightarrow] \arrow[d] & {[\mathbf{C}P_n^m, L_E\mathbf{S}]} \arrow[d] \\
\vdots & \vdots
\end{tikzcd}.$$

The inverse limit of the left tower is not quite the Ext term associated to the Hopf algebroid $(A,\Gamma)=(E^*(\mathbf{C}P^\infty),(E\wedge E)^*(\mathbf{C}P^\infty))$. This is due to the fact that

$$(E\wedge E\wedge E)^*(\mathbf{C}P^\infty)\overset{\simeq}{\leftarrow}\Gamma\otimes_A\Gamma$$

(the left-hand side is a completion of the right-hand side).

Similarly, the inverse limit of the tower on the right-hand side is not quite $[\mathbf{C}P_n^\infty, L_E\mathbf{S}]$. It does not include the phantom maps.

Coming back to the problem of determining the induced map of the $\mathbf{S}^1$-transfer on E_2-terms, we look at the cobar construction functor (cf. [25, Appendix 1])

$$B^n(M)=M\otimes_{E_*}(E_*E)^{\otimes_{E_*}n}$$

for an (E_*,E_*E)-comodule M.

Since E_*, E_*E, and $H^*(\mathbf{C}P^\infty)$ are concentrated in even dimensions, and since B is an exact functor on flat E_*-modules, we have a short exact sequence of $B(E_*,E_*E)$-modules

$$0\leftarrow B(E^*\Sigma^{-1}\mathbf{S}^{-1}))\leftarrow B(E^*(\mathbf{C}P_{-1}^\infty))\leftarrow B(E^*(\mathbf{C}P_0^\infty))\leftarrow 0.$$

If we denote by $Z^*(X)\subseteq B^*(X)$ the cycles under the cosimplicial differential d_1, we have a diagram as shown in Fig. 1. It follows from the snake lemma that the kernel of the top right map is the image under the snake map

$$Z^{*-1}(E^*\mathbf{S}^{-2})\to \mathrm{Ext}(\mathbf{C}P_0^\infty),$$

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
 & \mathrm{Ext}(\Sigma^{-1}\mathbf{S}^{-1}) \arrow[u] & \mathrm{Ext}(\mathbf{C}P_{-1}^\infty) \arrow[u] \arrow[l] & \mathrm{Ext}(\mathbf{C}P_0^\infty) \arrow[u] \arrow[l] & \\
 & Z^*(E^*\Sigma^{-1}\mathbf{S}^{-1}) \arrow[u] & Z^*(E^*\mathbf{C}P_{-1}^\infty) \arrow[u] \arrow[l] & Z^*(E^*\mathbf{C}P_0^\infty) \arrow[u] \arrow[l] & 0 \arrow[l] \\
0 & B^{*-1}(E^*\Sigma^{-1}\mathbf{S}^{-1}) \arrow[u, "d_1"] \arrow[l] & B^{*-1}(E^*\mathbf{C}P_{-1}^\infty) \arrow[u, "d_1"] \arrow[l] & B^{*-1}(E^*\mathbf{C}P_0^\infty) \arrow[u, "d_1"] \arrow[l] & 0 \arrow[l] \\
 & Z^{*-1}(E^*\Sigma^{-1}\mathbf{S}^{-1}) \arrow[u] & Z^{*-1}(E^*\mathbf{C}P_{-1}^\infty) \arrow[u] \arrow[l] & Z^{*-1}(E^*\mathbf{C}P_0^\infty) \arrow[u] \arrow[l] & 0 \arrow[l] \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Fig. 1. This diagram admits a snake map.

which is, by following through the diagram, the image of $d_1|_{E^*\{z^{-1}\}}$. Now if T is a p-compact torus, the transfer map in cohomotopy is simply the r-fold smash product of the map represented at the E_2-level by $d_1|_{E^*\{z^{-1}\}}$.

6.2. The map $\mathbf{S}^d \to BG^{\mathfrak{g}} \to \Sigma^r BT$

We will first study the effect of this map on rational cohomology. By Theorem 12, $H^*_{\mathbf{Q}_p}(BG) = H^*_{\mathbf{Q}_p}(BT)^{W(G)}$ is always a polynomial algebra.

Proposition 27. *For a p-compact group G with maximal torus T, the following diagram commutes*:

$$\begin{tikzcd} H^*_{\mathbf{Q}_p}(BT_+) \arrow[r, "proj"] \arrow[d, "\Sigma^{-r}t^*"] & H^*_{\mathbf{Q}_p}(BT_+)\,/\,(H^*_{\mathbf{Q}_p}(BG)) \arrow[r, equal, "Cor.\ 13"'] & H^*_{\mathbf{Q}_p}(G/T_+) \arrow[d] \\ H^*_{\mathbf{Q}_p}(\Sigma^{-r}BG^{\mathfrak{g}}) \arrow[r, "\cong"'] & H^*_{\mathbf{Q}_p}(BG_+)\{\tau\} \arrow[r, "\tau\mapsto\iota"'] & H^*_{\mathbf{Q}_p}(\mathbf{S}^{d-r}), \end{tikzcd}$$

where τ is the Thom class of $BG^{\mathfrak{g}}$, and ι is the generator in $H^*(\mathbf{S}^{d-r})$.

This allows us to compute the effect of the map

$$\mathbf{S}^d \to BG^{\mathfrak{g}} \to \Sigma^r BT$$

in cohomology (this is the bottom composition of maps in the diagram) by simply evaluating at the image in $H^*_{\mathbf{Q}_p}(BT_+)/(H^*_{\mathbf{Q}_p}(BG))$ of the fundamental class of G/T.

Proof of the proposition. By the construction of the transfer map, we have a commutative diagram

$$\begin{tikzcd} G_+ \wedge_T \mathbf{S}^r \arrow[r] & \Sigma^r BT_+ \\ \mathbf{S}^d \arrow[r] \arrow[u] & BG^{\mathfrak{g}} \arrow[u] \end{tikzcd},$$

and by Theorem 2(2), the left-hand map $\mathbf{S}^d \to \Sigma^r G/T_+$ is an isomorphism in the top homology group. Desuspending r times and applying $H^*_{\mathbf{Q}_p}$ yields the commutativity of the diagram of the proposition. $\square$

Now let E be a $H\mathbf{Z}/p$-local complex oriented torsion free cohomology theory. Denote by $E_{\mathbf{Q}_p}$ the cohomology theory $X \mapsto E^*(X) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

We have $E^*(\mathbf{C}P^\infty) = E^*[\![z]\!] \hookrightarrow E^*_{\mathbf{Q}_p}(\mathbf{C}P^\infty)$, and the same is true for $E \wedge E$. Hence for computing a cobar representative, we can work with rational coefficients and always hope that in the end of our computations, everything turns out integral. To compute this, we can use the Chern characters

$$\exp : E^*_{\mathbf{Q}_p}(X) \to H^*_{\mathbf{Q}_p}(X)\hat{\otimes} E^*$$

and

$$\exp : (E \wedge E)^*_{\mathbf{Q}_p}(X) \to H^*_{\mathbf{Q}_p}(X)\hat{\otimes}\pi_*(E \wedge E)$$

which, for $X = \mathbf{C}P^\infty_+$, is the exponential map for the formal group law associated with E and an isomorphism, and for $X = BT_+$, a tensor power thereof. The smash product is formed in the $H\mathbf{Z}/p$-local category, as always.

This induces an isomorphism of $(E_{\mathbf{Q}_p *}, (E \wedge E)_{\mathbf{Q}_p *})$-modules

$$B(\exp)\colon B(E^*_{\mathbf{Q}_p} BT_+) \to B(E^*) \hat{\otimes} H^*_{\mathbf{Q}_p}(BT_+).$$

We have a commutative diagram

$$\begin{tikzcd}
 & B(E^*_{\mathbf{Q}_p} BT_+) \arrow[dl] \arrow[dd] & & B(E^*BT_+) \arrow[ll, tail] \arrow[dr] \arrow[dd] & \\
B(E^*_{\mathbf{Q}_p}) & & & & B(E^*) \arrow[llll, tail] \\
 & B(E^*_{\mathbf{Q}_p}) \otimes H^*(BT_+) \arrow[ul] & & B(E^*) \otimes H^*(BT_+) \arrow[ll, tail] \arrow[ur] &
\end{tikzcd}$$

So, to evaluate the class in $B(E^*(BT))$ computed in the first part, we apply $B(\exp)$ to it and obtain a class in $B(E^*) \otimes H^*_{\mathbf{Q}_p}(BT)$, which we then evaluate at the image of the fundamental class $[G/H]$ in $(H_{\mathbf{Q}_p})_{d-r}(G/H_+) \to (H_{\mathbf{Q}_p})_{d-r}(BT_+)$. This class then must actually be integral.

7. The family No. 3 of groups μ_m

The p-compact group μ_m, for $p \equiv 1$ (m), has rank 1 and Weyl group $W \leqslant \mathbf{Z}_p^\times$ a cyclic subgroup of order m of the p-adic units. It is a nonmodular group and can therefore be constructed as

$$B\mu_m = L_p(K(\mathbf{Z}_p, 2) \times_W EW).$$

Therefore, $H^*(B\mu_m; \mathbf{Z}_p) = \mathbf{Z}_p[\![z]\!]^W$, where a W acts on z by multiplication. This shows that

$$H^*(B\mu_m; \mathbf{Z}_p) = \mathbf{Z}_p[\![z^m]\!] \hookrightarrow \mathbf{Z}_p[\![z]\!].$$

The fundamental class

$$[\mu_m/T] \in H^*_{\mathbf{Q}_p}(BT)/(H^*_{\mathbf{Q}_p}(B\mu_m)) = \mathbf{Q}_p[z]/(z^m)$$

is z^{m-1}, and we conclude that μ_m has dimension $1 + 2(m-1)$. It is straightforward to see that for $m < p-1$, μ_m cannot represent a nontrivial homotopy class in the p-stems because $(\pi_n^s)_{(p)} = 0$ for $0 < n < 2p-3$. But $(\pi^s_{2p-3})_{(p)} = \mathbf{Z}/p\{\alpha_1\}$, and we will see that μ_{p-1} represents this class.

Recall [1, Chapter 2] that $BP_*BP = BP_*[t_i]$ with $|t_i| = 2p^i - 2$. It follows from the formulas in [1, II.16] or [25, Appendix 2.1] that in the p-completed BP-spectral sequence associated to the Hopf algebroid (6.1) for $\pi^*(\mathbf{C}P^\infty_{-1})$,

$$(\eta_R(z))^{-1} = \sum_{i \geqslant 0}^{F} t_i z^{p^i},$$

where the sum is the formal group sum associated to the universal formal group law of BP and the inverse is with respect to composition of power series. Thus,

$$\eta_R(z) = z - t_1 z^p + O(z^{p+1}), \text{ hence } \eta_R(z^{-1}) = z^{-1} + t_1 z^{p-2} + O(z^{p-1}).$$

Applying the Chern character to this power series does not change it up to $O(z^{p-1})$, and hence $[\mu_{p-1}]$ is the coefficient of z^{p-2} of this series (up to a sign), which is t_1. Lying in filtration 1, t_1 represents the homotopy class α_1 [25].

8. Some exceptional cases

8.1. The 5-compact group No. 8

The pseudo-reflection group G which is No. 8 in Shephard and Todd's list has order 96 and is generated, as a complex reflection group, by the two reflections

$$\begin{pmatrix} -\mathrm{i} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{2}-\frac{\mathrm{i}}{2} & -\frac{1}{2}+\frac{\mathrm{i}}{2} \\ \frac{1}{2}-\frac{i}{2} & \frac{1}{2}-\frac{i}{2} \end{pmatrix}.$$

The ring of invariants $\mathbf{Z}_5[x_1,x_2]^G$ is polynomial because 5 does not divide the order of G; a straightforward calculation shows that it is generated by the polynomials

$$\mu = x_1^8 + 14x_1^4x_2^4 + x_2^8$$

and

$$v = x_1^{12} - 33x_1^8x_2^4 - 33x_1^4x_2^8 + x_2^{12}.$$

Hence $H^*(BG;\mathbf{Z}_5) = \mathbf{Z}_5[\mu, v]$, and by Proposition 14 the cohomology of G/T is given by

$$H^*(G/T_+;\mathbf{Z}_5) = H^*(BT_+;\mathbf{Z}_5)/(H^*(BG;\mathbf{Z}_5)).$$

A Gröbner basis calculation shows that the top class in $H^{36}(G/T;\mathbf{Z}_5)$ is

$$x_1^7x_2^{11} = -x_1^{11}x_2^7 = -\tfrac{1}{13}x_1^3x_2^{15} = \tfrac{1}{13}x_1^{15}x_2^3. \tag{8.1}$$

We will use the 5-primary BP-spectral sequence for $\pi^*(\mathbf{C}P_{-1}^{\infty})$ to determine the homotopy class G represents. In this spectral sequence,

$$\begin{aligned} \eta_R(z) &= z - t_1z^5 + (5t_1^2 + t_1v_1)z^9 + (-35t_1^3 - 12t_1^2v_1 - t_1v_1^2)z^{13} \\ &\quad + (285t_1^4 + 137t_1^3v_1 + 21t_1^2v_1^2 + t_1v_1^3)z^{17} + O(z^{21}) \end{aligned}$$

and hence

$$\begin{aligned} d_1(z^{-1}) &= -(t_1z^3) + (4t_1^2 + t_1v_1)z^7 + (-26t_1^3 - 10t_1^2v_1 - t_1v_1^2)z^{11} \\ &\quad + (204t_1^4 + 106t_1^3v_1 + 18t_1^2v_1^2 + t_1v_1^3)z^{15} \\ &\quad + O(z^{19}) \in (BP\wedge BP)^*(\mathbf{C}P_+^{\infty}). \end{aligned}$$

Applying the Chern character to this class yields

$$f(z) = -t_1 z^3 + \left(4t_1^2 + \frac{8t_1v_1}{5}\right) z^7 + \left(-26t_1^3 - \frac{78t_1^2v_1}{5} - \frac{78t_1v_1^2}{25}\right) z^{11}$$
$$+ \left(204t_1^4 + \frac{816t_1^3v_1}{5} + \frac{1224t_1^2v_1^2}{25} + \frac{816t_1v_1^3}{125}\right) z^{15}$$
$$+ O(z^{19}) \in BP_*BP \otimes H^*_{\mathbf{Q}_p}(\mathbf{C}P^\infty_+).$$

We need to evaluate the class $f(z) \otimes f(z)$ at the classes given in (8.1) and add them up. This yields that $[G]$ is represented in dimension 2 of the cobar complex by

$$-204t_1 \otimes t_1^4 - 808t_1^2 \otimes t_1^3 - 1208t_1^3 \otimes t_1^2 - 604t_1^4 \otimes t_1$$
$$-160v_1t_1 \otimes t_1^3 - 480v_1t_1^2 \otimes t_1^2 - 320v_1t_1^3 \otimes t_1$$
$$-48v_1^2t_1 \otimes t_1^2 - 48v_1^2t_1^2 \otimes t_1.$$

By adding a suitable boundary, namely

$$d_1(4t_1^5 + 45v_1t_1^4 + 34v_1^2t_1^3 + 10v_1^3t_1^2 + v_1^4t_1),$$

we see that this class is homologous to

$$t_1 \otimes t_1^4 + t_1^4 \otimes t_1 + 2(t_1^2 \otimes t_1^3 + t_1^3 \otimes t_1^2),$$

which is a representative of β_1 in the ANSS.

8.2. The 3-compact group Za_2 (No. 12)

The Weyl group W of the modular group Za_2 constructed by Zabrodsky [31] is generated by the two matrices

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{2} + \frac{i}{2} & \frac{1}{2} + \frac{i}{2} \\ -\frac{1}{2} + \frac{i}{2} & \frac{1}{2} - \frac{i}{2} \end{pmatrix}.$$

Although $3 | \#W$, the ring of invariants $\mathbf{Z}_3[x_1, x_2]^W$ is polynomial, generated by the polynomials

$$\mu = x_1^8 + 14x_1^4x_2^4 + x_2^8$$

and

$$v = x_1^5x_2 - x_1x_2^5.$$

We find that the top class in $H^{36}(G/T_+; \mathbf{Z}_3)$ is

$$x_1^4x_2^8 = x_1^8x_2^4 = -\tfrac{1}{15}x_1^{12} = \tfrac{1}{15}x_2^{12}. \tag{8.2}$$

In the 3-primary BP-ANSS, logarithm, exponential, and universal isomorphism are all odd power series; hence, evaluation at the class above yields 0 without further computations.

This means that $[Za_2]$ has filtration at least 3; however, the only class in dimension 38 in the Adams–Novikov E_2-term is $\beta_{3/2}$ in filtration 2.

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